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TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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No. 1032

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HEAT TRANSFER IN THE TURBULENT BOUNDARY LAYER  
OF A COMPRESSIBLE GAS AT HIGH SPEEDS

By F. Frankl

and

FRICITION IN THE TURBULENT BOUNDARY LAYER  
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By F. Frankl and V. Voishel

Central Aero-Hydrodynamical Institute

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Washington  
October 1942

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HEAT TRANSFER IN THE TURBULENT BOUNDARY LAYER

OF A COMPRESSIBLE GAS AT HIGH SPEEDS<sup>1</sup>

By F. Frankl

SUMMARY

The Reynolds law of heat transfer from a wall to a turbulent stream is extended to the case of flow of a compressible gas at high speeds. The analysis is based on the modern theory of the turbulent boundary layer with laminar sublayer. The investigation is carried out for the case of a plate situated in a parallel stream. The results are obtained independently of the velocity distribution in the turbulent boundary layer. The heat transfer  $q$  in unit time is a function of the frictional

coefficient  $\psi = \frac{2T}{\rho u^2}$ , the free stream temperature  $T$ ,

the temperature at the plate  $T_*$ , the free-stream velocity  $U$  and the density  $\rho$ , the velocity at the boundary of the laminar sublayer and the basic turbulent layer  $u_1$ , and the Prandtl number for the given gas

$$Pr = \frac{c_p \mu}{\lambda}.$$

The problem of heat transfer at high velocities has at the present time become a very pressing one both in connection with heat technology and high-speed aviation. Recently this problem has been encountered in the computation of the surface of a wing radiator. The existing papers on heat transfer from the wall to the gas in part refer to small speeds (Reynolds, Prandtl, and others) while with regard to high-speed flow some (Buseman, Frankl) consider only the laminar regime, others (Crocco) neglect the laminar sublayer in the turbulent layer, while still others (Guchman, Shirokov) start out from

<sup>1</sup>Report No. 240, of the Central Aero-Hydrodynamical Institute, Moscow, 1935.

relatively rough hydraulic assumptions. In the present paper the problem is treated from the point of view of the modern turbulence theory.

We shall consider the case of plane parallel flow and in particular the boundary layer of a plate in a stream parallel to it. We shall first review some basic known results.

The instantaneous velocity components  $u$ ,  $v$  and the values of the state variables  $p$ ,  $\rho$ ,  $T$  are broken up into their mean values  $\bar{u}$ ,  $\bar{v}$ , ... and fluctuations about the mean values  $u'$ ,  $v'$ , ... We assume that the turbulent fluctuations of the density  $\rho'$  are small by comparison with the mean density  $\bar{\rho}$ . We then have the following equations for the turbulent stresses  $\tau_{xx}$ ,  $\tau_{xy}$ ,  $\tau_{yy}$  and the heat transfer per unit time through unit area  $q_x$ ,  $q_y$ :

$$\left. \begin{aligned} \tau_{xx} &= \bar{\rho} u'^2 \\ \tau_{xy} &= \bar{\rho} u' v' \\ \tau_{yy} &= \bar{\rho} v'^2 \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} q_x &= c_p \bar{\rho} u' T' \\ q_y &= c_p \bar{\rho} v' T' \end{aligned} \right\} \quad (2)$$

where the bars denote the time-averaged values, and the equations of motion

$$\left. \begin{aligned} \rho \frac{d\bar{u}}{dt} &= - \frac{\partial \bar{p}}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} \\ \rho \frac{d\bar{v}}{dt} &= - \frac{\partial \bar{p}}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} \end{aligned} \right\} \quad (3)$$

$$\frac{\partial(\bar{\rho}u)}{\partial x} + \frac{\partial(\bar{\rho}v)}{\partial y} = 0 \quad (4)$$

$$\bar{\rho} \frac{d}{dt} \left( J c_p \bar{T} + \frac{\bar{u}^2 + \bar{v}^2}{2} + \frac{\bar{u}'^2 + \bar{v}'^2}{2} \right)$$

$$= - \frac{\partial}{\partial x} (\bar{u} \tau_{xx} + \bar{v} \tau_{xy}) - \frac{\partial}{\partial y} (\bar{u} \tau_{xy} + v \tau_{yy}) - J \left[ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right] \quad (5)$$

The viscosity forces are here neglected by comparison with the turbulent friction.

We now proceed to the case of the plate. In this case  $\bar{p}$  has a constant value. We choose the edge of the plate for the  $x$ -axis. We may then neglect the derivatives with respect to  $y$ , and  $\bar{v}$  by comparison with  $\bar{u}$ . We assume, finally, that the average energy of the fluctuating turbulent velocities has a constant value. Equations (3) and (5) then become

$$\rho \frac{d\bar{u}}{dt} = - \frac{\partial \tau_{xy}}{\partial y} \quad (3')$$

$$\bar{p} \frac{d}{dt} \left( J c_p T + \frac{\bar{u}^2}{2} \right) = - \frac{\partial}{\partial y} (\bar{u} \tau_{xy}) - J \frac{\partial q_y}{\partial y} \quad (5')$$

For the determination of  $\tau_{xy}$  and  $q_y$  we start from the following generally accepted assumptions. The deviating of a body of fluid arriving at the point  $(xy)$  from the point with ordinate  $y + l_y$ , is given by

$$u' = l_y \frac{du}{dy} \quad (6)$$

Similarly we obtain

$$T' = l_y \frac{\partial T}{\partial y} \quad (7)$$

Substituting these expressions in (1) and (2) we obtain

$$\tau_{xy} = \bar{p} v' l_y \frac{\partial \bar{u}}{\partial y} = A \frac{\partial \bar{u}}{\partial y} \quad (8)$$

$$q_y = c_p \bar{p} v' l_y \frac{\partial \bar{T}}{\partial y} = \Lambda \frac{\partial \bar{T}}{\partial y} \quad (9)$$

where  $A$  is the coefficient of turbulent apparent viscosity,  $\Lambda$  the coefficient of turbulent heat conductivity. From the equations

$$A = \bar{p} v' l_y$$

$$\Lambda = c_p \bar{p} v' l_y$$

there follows directly the well-known equation of Prandtl

$$\Lambda = c_p A \quad (10)$$

Substituting equations (8) and (9) in equations (3') and (5') and dropping the averaging signs, since in what follows we shall deal only with mean values, we obtain

$$\rho \frac{du}{dt} = - \frac{\partial}{\partial y} \left( A \frac{\partial u}{\partial y} \right) \quad (11)$$

$$\rho \frac{d}{dt} \left( J c_p T + \frac{u^2}{2} \right) = - \frac{\partial}{\partial y} \left[ A \frac{\partial}{\partial y} \left( J c_p T + \frac{u^2}{2} \right) \right] \quad (12)$$

Thus the velocity and total energy  $J c_p T + \frac{u^2}{2}$

satisfy the same linear differential equation. All boundary conditions may therefore be satisfied by setting

$$J c_p T + \frac{u^2}{2} = au + b \quad (13)$$

where  $a$  and  $b$  are independent of  $y$ . This relation was first derived by Crocco.

We now proceed with our problem, requiring from the above relations only equations (10) and (13). We shall use the following notation:

$$e = J c_p T + \frac{u^2}{2} \quad \text{total energy}$$

$\bar{u}$ ,  $\bar{T}$ ,  $\bar{e}$ ,...the values of  $u$ ,  $T$ ,  $e$ ,...in the free flow  
 $u_1$ ,  $T_1$ ,  $e_1$ , the values at the boundary of the laminar sublayer and turbulent layer,  $T_*$ ,  $\rho_*$ ,  $\mu_*$  the values of  $T$ ,  $\rho$ ,  $\mu$  at the wall.

Equation (13) may then be rewritten in the form

$$e - e_1 = \frac{\bar{e} - e_1}{\bar{u} - u_1} (u - u_1) \quad (14)$$

In the laminar sublayer the tangential stress  $\tau_{xy}$  and the heat transfer  $q_y$  are determined by the formulas

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}$$

$$q_y = \lambda \frac{\partial T}{\partial y}$$

where  $\mu$  is the coefficient of viscosity and  $\lambda$  the coefficient of heat conductivity. From the physical significance of  $\tau_{xy}$  and  $q_y$  it follows that they should be continuous over the entire region of flow. (The schematic representation of completely laminar and completely turbulent sublayers leads therefore to a dis-

continuity of  $\frac{\partial u}{\partial y}$  and  $\frac{\partial T}{\partial y}$  in passing from the laminar to the turbulent layer since  $\mu$  is considerably less than  $\Lambda$  and  $\lambda$  less than  $\Lambda$ ).

As a first approximation we assume a linear distribution of velocity and temperature in the laminar sublayer. We then have

$$u_1 = \frac{\tau}{\mu_*} h \quad (15)$$

$$T_1 - T_* = \frac{q}{\lambda_*} h \quad (16)$$

where  $\tau$  and  $q$  are the values of  $\tau_{xy}$  and  $q_y$  directly at the wall and  $h$  is the thickness of the laminar sublayer. From the assumption of the continuity of  $\tau_{xy}$  and  $q_y$  it then follows that

$$\left. \Lambda_1 \frac{\partial u}{\partial y} \right|_1 = \tau \quad (17)$$

$$\left. \Lambda_1 \frac{\partial T}{\partial y} \right|_1 = q \quad (18)$$

From equations (10), (17), and (18) we obtain the relation

$$\left. \frac{\partial T}{\partial y} \right|_1 = \frac{q}{c_p \tau} \left. \frac{\partial u}{\partial y} \right|_1 \quad (19)$$

We now differentiate equation (14) with respect to  $y$  for  $y = h$  and obtain

$$\left. \frac{\partial e}{\partial y} \right|_1 = \frac{\bar{e} - e_1}{\bar{u} - u_1} \left. \frac{du}{dy} \right|_1 \quad (20)$$

Substituting the value  $\left. \frac{\partial T}{\partial y} \right|_1$  from (18) we obtain

$$u_1 + \frac{Jq}{\tau} = \frac{\bar{e} - e_1}{\bar{u} - u_1} \quad (21)$$

Finally substituting the value of  $T_1$ , from (15) and (16)

$$T_1 = T_* + \frac{q}{\lambda_*} \frac{\mu_*}{\tau} u_1 \quad (22)$$

there is obtained the following relation between  $q$ ,  $\tau$ ,  $\bar{u}$ ,  $T$ ,  $T_*$  and  $u_1$ :

$$u_1 + \frac{Jq}{\tau} = \frac{J c_p \left( \bar{T} - T_* - \frac{q}{\lambda_*} \frac{\mu_*}{\tau} u_1 \right) + \frac{\bar{u}^2}{2} - \frac{u_1^2}{2}}{\bar{u} - u_1}$$

Solving this equation for  $q$  we obtain

$$q = \tau \frac{c_p (\bar{T} - T_*) + \frac{(\bar{u} - u_1)^2}{2\tau}}{\bar{u} + u_1 (Pr - 1)} \quad (23)$$

or

$$q = c_p \frac{\psi_*}{2} \rho_* \bar{u} \frac{\bar{T} - T_* + \frac{(\bar{u} - u_1)^2}{2J c_p}}{1 + (Pr - 1) \frac{u_1}{\bar{u}}} \quad (24)$$

where  $\Pr = \frac{c_p \mu}{\lambda}$  ( $= 0.8$  for diatomic gases).<sup>1</sup>

The heat transfer through unit area per unit time can therefore be found when the temperature in the free flow and at the wall, the velocity of the flow, the coefficient of friction, and the density at the wall are known. The velocity  $u_1$  on the right side of the equation can be determined in the following manner. In an incompressible fluid according to the hypothesis of Kármán we have

$$u_1 = s \sqrt{\frac{\tau}{\rho}} \quad (25)$$

where  $s$  is an absolute constant. According to the tests of Nikuradse it has the value

$$s = 11.6 \quad (26)$$

In a compressible gas we have the analogous relation

$$u_1 = s \sqrt{\frac{\tau}{\rho_*}} \quad (25')$$

where  $s$  may in general depend also on the nondimensional parameters  $\frac{\tau}{\rho}$  and  $\frac{q \mu_*}{\lambda_* T_* \sqrt{\tau \rho_*}}$ .

Since, however,  $s$  enters only into terms that are comparatively small at not too high (in particular, subsonic) velocities we may as a first approximation use the value  $s = 11.6$ . From (25') we have

$$\frac{u_1}{u} = s \sqrt{\frac{\psi_*}{2}} \quad (27)$$

so that equation (29) may be written also in the form

<sup>1</sup>In all our formulas  $c_p$  denotes the heat capacity of a unit mass at constant pressure. If unit weight instead of unit mass is used then  $c_p$  is replaced by  $g c_p$ .

$$q = c_p \frac{\psi_*}{3} \rho_* \bar{u} \frac{\bar{T} - T_* + \left(1 - s\sqrt{\frac{\psi_*}{2}}\right)^2 \frac{\bar{u}^2}{2Jc_p}}{1 + (Pr - 1)s\sqrt{\frac{\psi_*}{2}}} \quad (23)$$

## CORRECTION

These papers had already gone to press when it was pointed out to the author by N. Shirokov that according to the unpublished results obtained by him it is not correct to neglect the dissipation of energy in the laminar sublayer. As a result the curve of temperature distribution in the laminar sublayer cannot be considered a straight line. Instead is obtained the parabola

$$T = T_* - \frac{q}{\lambda_*} y - \frac{\tau^2}{2J\mu_*\lambda_*} y^2$$

Equation (23) above is therefore replaced by the equation

$$q = \tau \frac{c_p (T_* - \bar{T}) - \frac{\bar{u}^2 + (Pr - 1)u_1^2}{2J}}{\bar{u} + (Pr - 1)u_1}$$

and similarly equation (12) in the second article is replaced by the equation

$$\frac{1}{\sigma} = 1 + \frac{\theta}{Pr} [\varphi + s(Pr - 1)] - \frac{k - 1}{2k} \nu [\varphi^2 - (Pr - 1)s^2].$$

Correspondingly there is a change in the remaining conclusions of our articles: in particular, there is a quantitative change in the dependence of the friction coefficient on  $Ba$ . We may also note that A. Guchman and his coworkers (Leningrad) found experimentally a considerable lowering in the friction coefficient in pipes with increasing  $Ba$ .

FRICITION IN THE TURBULENT BOUNDARY LAYER  
OF A COMPRESSIBLE GAS AT HIGH SPEEDS

By F. Frankl and V. Voishel

SUMMARY

The Prandtl-Kármán theory of turbulent friction, the so-called "Mischungsweg" theory, is extended to cover the case of high-speed motion of a compressible gas with heat transfer from the wall. A correction of the Kármán friction law is derived that depends on the Bairstow number (the compressibility effect) and on the ratio of the absolute temperatures in the free stream and at the wall. The analysis is based on the heat transfer law developed in a preceding article by F. Frankl.

The airfoil drag is made up of two parts, namely, the drag of a plate of the same chord and the "form drag," that is, the excess drag above that of the plate. The effect of the compressibility on these two drag components depends on a number of factors. The form drag coefficient varies mainly with change in the velocity distribution about the airfoil while the plate drag coefficient depends only on the change in the friction coefficient. In the present paper we shall investigate the effect of the compressibility on the frictional coefficient and also the effect of heating the wall, having in mind the case of the wing radiator. We restrict ourselves to the case of the turbulent boundary layer of a plate situated in a flow of constant velocity at zero angle of attack. For this purpose we generalize the new theory of the turbulent boundary layer, the so-called mixing path (Mischungsweg) theory.

Before writing down the fundamental differential equations we pass to nondimensional magnitudes by introducing measures for the lengths, velocities, and gas state variables. The nondimensional velocities and state variables as functions of nondimensional coordinates will then depend on parameters which are nondimensional combinations of the measures, the boundary conditions at the plate, and the physical constants of the gas. As boundary data we may take the frictional stress  $\tau$  the heat transfer per unit time  $q$  and the state variables of the gas  $p$ ,  $p_*$ , and  $T_*$  at the plate. The pressure  $p$ , as

usual in the boundary layer theory, is assumed constant. We now introduce  $\rho_*$  and  $T_*$  as measures for  $\rho$  and  $T$ . As a measure of the velocity we shall take the "friction velocity"

$$v_* = \sqrt{\frac{\tau}{\rho_*}} \quad (1)$$

and as measure of the length

$$\frac{\mu_*}{\rho_* v_*} = \frac{\mu_*}{\sqrt{\rho_* \tau}} \quad (2)$$

where  $\mu_*$  is the value of the viscosity coefficient at the plate. By introducing the above measures the parameters of the solution may now be only nondimensional combinations of the boundary data and the physical constants.

Together with Kármán we assume that only the end conditions at points very near the wall produce an effect.

In the absence of compressibility and heat transfer such combinations do not exist, as a result of which fact there is obtained as is known a universal velocity distribution not dependent on any parameters. We shall therefore in what follows denote these measures as "universal." The only such independent parameters are

$$\gamma = \frac{\tau}{p} \quad (3)$$

$$\theta = \frac{q\mu_*}{\lambda_* T_* \sqrt{\rho_* \tau}} \quad (4)$$

and the magnitudes depending only on the physical constants

$$k = \frac{c_p}{c_v} \quad (5)$$

$$Pr = \frac{c_p \mu}{\lambda} \quad (6)$$

For diatomic gases we have  $k = 1.4$  and the Prandtl number  $Pr = 0.8$ . (In the above  $c_p$ ,  $c_v$  are the specific heats at constant pressure and volume, respectively,  $\lambda$  the heat conduction coefficient of the gas,  $\lambda_*$  the value of  $\lambda$  at the wall.)

We shall now investigate the turbulent boundary layer of the plate. We shall take the edge of the plate for the  $x$ -axis; the variable velocity in the boundary layer will be denoted by  $u$ ; the other notation is the same as for the preceding article "Heat Transfer".

We now assume that for the compressible gas the fundamental differential equation of the Kármán turbulence theory remains valid:

$$\kappa^2 p \frac{\left(\frac{\partial u}{\partial y}\right)^4}{\left(\frac{\partial^2 u}{\partial y^2}\right)^2} = \tau \quad (7)$$

where  $\kappa$  is an absolute constant equal to 0.4 and  $\tau$  is the frictional stress at a distance  $y$  from the plate. When the coordinate  $x$  (that is, the distance from the forward edge of the plate) is sufficiently large, however, we may neglect the forces of inertia. Because of this and from the constancy of the pressure  $p$  it follows that  $\tau$  does not depend on  $y$ .

We now proceed to the "universal" measures and introduce the nondimensional magnitudes:

$$\eta = \frac{y \sqrt{\rho_* \tau}}{\mu_*} \quad (8)$$

$$\varphi = \frac{u}{v_*} \quad (9)$$

$$\sigma = \frac{p}{p_*} = \frac{T_*}{T} \quad (10)$$

Equation (7) then assumes the form

$$\kappa^2 \sigma = \frac{\left(\frac{\partial^2 \varphi}{\partial \eta^2}\right)^2}{\left(\frac{\partial \varphi}{\partial \eta}\right)^4} \quad (11)$$

In order to obtain the differential equation for  $\varphi$  it remains only to express  $\sigma$  in terms of  $\varphi$ . This is possible with the aid of equations (14) and (21) of the preceding article. From these there is obtained an equation analogous to equation (23) of the preceding article:

$$q = \tau \frac{c_p (T - T_*) + \frac{(u - u_1)^2}{2Jc_p}}{u + u_1 (Pr - 1)}$$

which after substituting the universal measures is reduced to the form

$$\frac{1}{\sigma} = 1 + \frac{\theta}{Pr} \left[ \varphi + s (Pr - 1) \right] - \frac{k - 1}{2k} \gamma (\varphi - s)^2 \quad (12)$$

where  $s = u_1/v_*$  according to the above considerations can depend only on  $\gamma$ ,  $\theta$ ,  $k$ , and  $Pr$ .<sup>1</sup>

Substituting expression (12) in (11) we obtain the differential equation for  $\varphi$  which is easily solved in terms of elementary functions.

To determine the boundary conditions we start from the hypothesis of linear distribution of  $u$  in the laminar sublayer. This gives

$$u = \frac{\tau}{\mu_*} y$$

or

$$\varphi = \eta \quad (13)$$

and hence  $\varphi_1 = \varphi(s) = s$ .

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<sup>1</sup>See Correction on p. 8.

As regards the value of  $\frac{\partial \phi}{\partial \eta}$  in the turbulent layer for  $y = s$  (that is at the boundary of the laminar sublayer) we have

$$\left. \frac{\partial \phi}{\partial \eta} \right|_{\eta = s} = f \quad (14)$$

where  $f$ , like  $s$ , can depend only on  $\gamma$ ,  $\epsilon$ ,  $k$  and  $Pr$ . In an incompressible fluid in the absence of heat transfer  $f$  has the constant value

$$f = 0.289 \quad (15)$$

Our problem now is to find the coefficient of friction  $\psi_*$  (see preceding article) as a function of the Bairstow number, the ratio of the temperature in the free stream to that at the wall and the Reynolds number at a given point of the plate, that is.

$$\psi_* = \psi_* \left( \xi, Ba_*, \frac{T_*}{T} \right)$$

$$\xi = \frac{\rho_* \bar{u}_x}{\mu_*} \quad (16)$$

where

$$\left. \begin{aligned} Ba_* &= \frac{\bar{u}}{a_*} \\ a_* &= \sqrt{(k-1) J c_p T_*} \end{aligned} \right\} \quad (17)$$

(=  $a_*$  velocity of sound at the wall)

Instead of  $T_*/T$  it is convenient in what follows to introduce the parameter  $\omega$  determined by the equation

$$\frac{T_*}{T} = \frac{k-1}{2} Ba_*^2 \omega + 1 \quad (18)$$

In what follows we shall assume that  $\omega$  is of the order of magnitude of unity. The physical significance

of this assumption lies in the fact that  $T_* - \bar{T}$  is of the same order of magnitude as the temperature difference that occurs with friction and without heat transfer. For wing radiators at large velocities this assumption may be considered as satisfied.

It is now necessary to determine  $\psi_*$  in the form

$$\psi_* = \psi_*(\xi, Ba_*, \omega)$$

For this purpose it is necessary first of all to express the parameters  $\gamma$  and  $v$  in the velocity distribution law in terms of  $\psi_*$ ,  $Ba_*$  and  $\omega$ .

For  $\gamma$  we obtain immediately

$$\gamma = \frac{k}{2} \psi_* Ba_*^2 \quad (19)$$

and for  $\phi$  from equation (28) of the preceding article

$$\phi = \frac{\Pr \sqrt{\frac{\psi_*}{2}}}{1 + (\Pr - 1) \bar{s} \sqrt{\frac{\psi_*}{2}}} \frac{k - 1}{2} Ba_*^2 \left[ \omega - \left( 1 - \bar{s} \sqrt{\frac{\psi_*}{2}} \right)^2 \right] \quad (20)$$

We now substitute (19) and (20) in the solution of equations (11) and (12) for the boundary conditions (13) and (14). If this solution is expanded in a Taylor series in powers of  $Ba_*^2$  and only the zero and first power terms retained there is obtained

$$\eta = s + \frac{e^{\kappa(\phi - s)} - 1}{\kappa f} - \frac{k - 1}{4\kappa^2 f} Ba_*^2 \psi_* \left\{ e^{\kappa(\phi - 1)} \left[ \frac{\kappa}{2} (m - \phi)^2 + n \right] - r \right\} \quad (21)$$

where  $m$ ,  $n$  and  $r$  depend only on  $\omega$ ,  $\Pr$ ,  $\psi_*$ ,  $k$ , and  $s$ . In the case  $Ba_* = 0$  we obtain the well-known logarithmic velocity distribution.

For the solution of this problem it is still required that the velocity distribution in the well-known integral condition of Kármán be substituted:

$$\frac{d}{dx} \int_0^s \rho u (u - \bar{u}) dy = -\tau$$

or in absolute measure

$$d\xi = -\frac{2}{\psi_*} d \left\{ \int_0^{\bar{\eta}} \sigma \varphi (\varphi - \bar{\varphi}) d\eta \right\} \quad (22)$$

where

$$\bar{\eta} = \frac{\partial \sqrt{\rho_* \tau}}{\mu_*}$$

Carrying out the integration of equation (22) on the assumption that  $f$  and  $s$  maintain the constant values  $s = 11.6$  and  $f = 0.289$  and setting

$$\kappa \sqrt{\frac{2}{\psi}} = z$$

we obtain

$$\begin{aligned} \xi = & P_1(z) + e^z P_2(z) - B a_*^2 \left\{ S_0(\omega) + S_1(\omega)z + S_2(\omega) \ln(z-c) \right. \\ & + C_1 \ln z + \frac{C_2 + S_3(\omega)z + S_4(\omega)z^2}{z(z-c)^2} + e^z \left[ S_5(\omega) \right. \\ & + S_6(\omega)z + S_7(\omega)z^2 + \frac{C_3 + S_8(\omega)z + S_9(\omega)z^2}{z(z-c)^2} \\ & \left. \left. + C_4 E_i(z) + S_{10}(\omega) E_i(z-c) \right] \right\} \quad (23) \end{aligned}$$

where

$$P_1(z) = 9542 - 3134z + 246z^2$$

$$P_2(z) = 1.31(z^2 - 4z + 6)$$

$$S_0(\omega) = 86217 - 2676(\omega - 1) + 168(\omega - 1)^2$$

$$S_1(\omega) = 129.7(\omega - 1) - 38.0(\omega - 1)^2$$

$$S_2(\omega) = -[25353 - 4567(\omega - 1) + 173(\omega - 1)^2]$$

$$S_3(\omega) = 373151 + 4380(\omega - 1) - 162(\omega - 1)^2$$

$$S_4(\omega) = [175174 + 6211(\omega - 1) - 223(\omega - 1)^2]$$

$$S_5(\omega) = -1.011 + 3.669(\omega - 1) - 0.055(\omega - 1)^2$$

$$S_6(\omega) = 1.003 - 0.965(\omega - 1) + 0.105(\omega - 1)^2$$

$$S_7(\omega) = 0.131 + 0.131(\omega - 1) - 0.049(\omega - 1)^2$$

$$S_8(\omega) = 86.187 - 1.723(\omega - 1) - 0.048(\omega - 1)^2$$

$$S_9(\omega) = [55.959 - 0.502(\omega - 1) - 0.097(\omega - 1)^2]$$

$$S_{10}(\omega) = -[144.71 + 3.26(\omega - 1) + 0.51(\omega - 1)^2]$$

$$C = 0.928$$

$$C_1 = 18433$$

$$C_2 = 186231$$

$$C_3 = -58.416$$

$$C_4 = 184.94$$

$$\Xi_1(z) = \int_{-\infty}^z \frac{e^z dz}{z}$$

We thus have an equation in the form  $\xi = \xi(\psi_*, Ba_*, \omega)$  and our fundamental problem is solved. It remains to note only that the dependence of the Reynolds number  $Re_g = \rho_* \bar{u} \delta / u_*$  on  $\psi_*$ ,  $Ba_*$ , and  $\omega$  can likewise be readily found.